Part IV

The Black-Scholes Framework

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The Black-Scholes Equation

Black-Scholes Model - An Overview

- In this section, we will derive the Black-Scholes formulas for option pricing.
- Fischer Black (1938 - 1995) and Myron Scholes (1941 -) formulated this model in *The Pricing of Options and Corporate Liabilities (1973)*, with Scholes winning the Nobel prize for his work.

Black-Scholes Model - An Overview

- We will follow the same framework that we have used for all the models we studied so far.
- First, we will define self-financing portfolios, and the notion of an arbitrage opportunity in this model. We then assume the Principle of No Arbitrage.
- The goal will be to prove the Risk-Neutral Pricing formula:

Theorem (Risk-Neutral Valuation)

Suppose the continuous-time model is arbitrage free. Then the risk-neutral price of a contingent claim $X = \Phi(S_T)$ at time $t \leq T$ is given by

$$\Pi_X(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right].$$

In particular,

$$\Pi_X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[X].$$

Black-Scholes Model - Setting

- Our market model is built on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.
- We assume that there is a Brownian Motion $\{W_t\}_{t\geq 0}$ that is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, and that $W_t - W_s$ is independent of $\mathcal{F}_s$ for $s < t$.
- This probability measure $\mathbb{P}$ is called the physical probability measure.
    - We have seen that in the binomial model, the physical measure $\mathbb{P}$ is not used in the pricing formula. This will be true for the Black-Scholes model as well.

Black-Scholes Model - Setting

- We assume that there is a risk-free asset $\{B_t\}_{t\geq 0}$ that earns continuous interest at a rate r. That is,

$$dB_t = rB_t\,dt\,.$$

- We will also assume that there is a risky stock (that does not pay dividends) $\{S_t\}_{t\geq 0}$ given by a GBM. That is,

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t^{\mathbb{P}}\,.$$

  It will be important in this chapter to specify which measure we are using. The random part of the SDE above is given by the $dW_t^{\mathbb{P}}$ term. This is a Brownian motion under $\mathbb{P}$. If we change the measure, then this may no longer be a Brownian motion.

Black-Scholes Model - Setting

- Since S_t is GBM, then we have all our nice results from before:

$$S_t \sim^{\mathbb{P}} log\mathcal{N}\left(\ln(S_0) + \left(\alpha - \frac{\sigma^2}{2}\right)t,\, \sigma^2 t\right),$$

$$S_t = S_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t^{\mathbb{P}}}.$$

Black-Scholes Model - Setting

- In a multiperiod model, we consider portfolios that we can rebalance every period in a self-financing manner.
- In continuous time, we take this to the extreme by assuming that we rebalance continuously. One could argue that this is not realistic in practice...

Definition (Portfolio Strategy)

A portfolio strategy is a stochastic process $\theta = \{\theta_t\}_{t\geq 0}$ where:

- For each t, we have $\theta_t = (\theta_t^S, \theta_t^B)$ is an adapted process.
- θ_t^S denotes the number of shares of stock held at time t for the next tiny interval dt.
- θ_t^B denotes the units of the risk-free asset at time t held for the next tiny interval dt.

Black-Scholes Model - Setting

- Note that this process needs to be adapted to make sense. In other words, at time t, we must know the portfolio composition θ_t. Otherwise, we would not be able to implement this strategy.

Definition (Value Process)

The value process of the portfolio strategy $\theta = \{\theta_t\}_{t \geq 0}$ is the stochastic process $\{V_t^\theta\}_{t \geq 0}$, where

$$V_t^\theta = \theta_t^S S_t + \theta_t^B B_t \, .$$

Black-Scholes Model - Setting

- The notion of a self-financing portfolio is more nuanced in this setting.
- Roughly, a portfolio strategy $\{\theta_t\}_{t\geq 0}$ is self-financing if

$$\theta^S_{t-dt} S_t + \theta^B_{t-dt} B_t = \theta^S_t S_t + \theta^B_t B_t \,.$$

- A complicated calculation (involving the matrix version of Ito's lemma) eventually yields the following:

Definition (Self-financing)

The portfolio strategy $\theta = \{\theta_t\}_{t\geq 0}$ is self-financing if

$$dV^\theta_t = \theta^S_t \, dS_t + \theta^B_t \, dB_t \,.$$

- The intuition here is more straightforward: the change in the portfolio value is only due to the change in value of the assets in the portfolio.

Black-Scholes Model - Setting

- However, the definition of an arbitrage opportunity is the same as before.

Definition (Arbitrage Opportunity)

An arbitrage opportunity is a self-financing portfolio strategy θ such that:

(i) $V_0^\theta \leq 0$, and,

(ii) At some $T > 0$, we have $\mathbb{P}(V_T^\theta \geq 0) = 1$ and $\mathbb{P}(V_T^\theta > 0) > 0$.

- The definition of attainability and completeness are also the same as before, with our updated notion of self-financing portfolio.

Black-Scholes Model - Derivation Framework

- The first step of developing the risk-neutral pricing formula is to derive the Black-Scholes Equation. This is a (stochastic) partial differential equation (PDE) that arbitrage-free prices Π_t must satisfy.
- The second step is to actually solve the Black-Scholes PDE. This will be given in the next section.
- We will take the following approach:
  - We will formulate an ansatz given all we have learned so far. This is an assumption about the form of the solution.
  - Using this ansatz, we will show that Π_t must satisfy the Black-Scholes PDE (step 1), and identify the solutions to the Black-Scholes PDE (step 2).
  - Once this is done, it is possible to reason backwards that our ansatz is indeed correct.

Developing the Black-Scholes PDE

Black-Scholes Model - Ansatz

- Consider a contingent claim of the form $X = \Phi(S_T)$. That is, the payoff is a function of the stock price at an expiry time T. Let Π_t denote the price of the contingent claim at time $t \in [0, T]$.
- As previously discussed, this continuous-time model can be seen as the limit of the multiperiod binomial model. Hence, we can use the binomial model as the basis of our ansatz.

Black-Scholes Model - Ansatz

- In fact, even writing down Π_t implicitly assumes that a (unique) price exists! Indeed, we would expect this to be true given what we know about the binomial model.

- Hence, we are assuming that the model is arbitrage-free and complete.

- Note that this is actually quite a strong assumption. In particular, completeness implies that the contingent claim can be replicated. Hence, we assume there exists a self-financing portfolio θ that replicates X:

$$\Pi_t = V_t^\theta = \theta_t^S S_t + \theta_t^B B_t\,.$$

- Our self-financing condition now implies that

$$dV_t^\theta = d\Pi_t = \theta_t^S\, dS_t + \theta_t^B\, dB_t\,.$$

Black-Scholes Model - Ansatz

- We have seen in the binomial model that $\Pi_T = \Phi(S_T)$. Also, we can identify the price of the contingent claim Π_t at each node.
- Furthermore, if we know the time t and the stock price S_t, then we know which node we are at. This suggests that Π_t is a function of time t and the stock price S_t.
- Hence, we will assume Π_t can be written as a function

$$\Pi_t = F(t, S_t) \, .$$

- Note that as a consequence of our assumptions, Π_t is an adapted process.

Black-Scholes Model - Developing the PDE

- The idea is that we now have two different representations of the dynamics of $V_t^\theta = \Pi_t = F(t, S_t)$, which must coincide.
- First, let us simplify the expression that we get from the self-financing condition:

$$
\begin{aligned}
d\Pi_t = dV_t^\theta &= \theta_t^S\, dS_t + \theta_t^B\, dB_t \\
&= \theta_t^S\, (\alpha S_t\, dt + \sigma S_t\, dW_t^{\mathbb{P}}) + \theta_t^B\, rB_t\, dt \\
&= (\theta_t^S \alpha S_t + \theta_t^B\, rB_t)\, dt + \theta_t^S \sigma S_t\, dW_t^{\mathbb{P}}\, .
\end{aligned}
$$

Black-Scholes Model - Developing the PDE

- Now applying Ito's lemma to the function $F(t, s)$, we have

$$
\begin{aligned}
dF(t,S_t) &= F_t(t,S_t)\,dt + F_s(t,S_t)\,dS_t + \frac{1}{2}F_{ss}(t,S_t)(dS_t)^2 \\
&= F_t(t,S_t)\,dt + F_s(t,S_t)\left(\alpha S_t\,dt + \sigma S_t\,dW_t^{\mathbb{P}}\right) \\
&\quad + \frac{1}{2}F_{ss}(t,S_t)\left(\alpha S_t\,dt + \sigma S_t\,dW_t^{\mathbb{P}}\right)^2 \\
&= F_t(t,S_t)\,dt + F_s(t,S_t)\left(\alpha S_t\,dt + \sigma S_t\,dW_t^{\mathbb{P}}\right) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t,S_t)\,dt \\
&= \left(F_t(t,S_t) + \alpha S_t F_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t,S_t)\right)dt + F_s(t,S_t)\sigma S_t\,dW_t^{\mathbb{P}}.
\end{aligned}
$$

Black-Scholes Model - Developing the PDE

- Putting these equations side-by-side, we have

$$d\Pi_t = (\theta_t^S \alpha S_t + \theta_t^B r B_t)\, dt + \theta_t^S \sigma S_t \, dW_t^{\mathbb{P}} \,,$$

$$d\Pi_t = \left(F_t(t, S_t) + \alpha S_t F_s(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t, S_t) \right) dt + F_s(t, S_t)\sigma S_t \, dW_t^{\mathbb{P}} \,.$$

- Since these SDEs represent the same thing, the drifts and the diffusions must be equal.

Black-Scholes Model - Developing the PDE

- From the diffusions, we see that

$$\theta_t^S = F_s(t, S_t)\,.$$

- Substituting this back into the value process equation gives

$$\theta_t^B B_t = V_t^\theta - \theta_t^S S_t = F(t, S_t) - S_t F_s(t, S_t)$$
$$\theta_t^B = F(t, S_t)/B_t - S_t F_s(t, S_t)/B_t\,.$$

- We have solved for the replicating portfolio strategy in terms of F. Note that the position the stock is the derivative of the option price with respect to the stock price.

Black-Scholes Model - Developing the PDE

- Finally, equating the drifts gives

$$\theta_t^S \alpha S_t + \theta_t^B r B_t = F_t(t, S_t) + \alpha S_t F_s(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t, S_t)$$

$$\alpha S_t F_s(t, S_t) + rF(t, S_t) - rS_t F_s(t, S_t) = F_t(t, S_t) + \alpha S_t F_s(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t, S_t)$$

$$rF(t, S_t) - rS_t F_s(t, S_t) = F_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t, S_t)$$

- Note that the terms involving α cancel out. This suggests that the option price does not depend on α.
- The equation we are left with is precisely the Black-Scholes equation.

Black-Scholes Model - The Black-Scholes PDE

Definition (Black-Scholes Equation)

For a given payoff function Φ, a function $F(t, S_t)$ satisfies the Black-Scholes equation if

$$\begin{aligned} F_t(t, S_t) + rS_t F_s(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t, S_t) &= rF(t, S_t)\,, \\ F(T, S_T) &= \Phi(S_T)\,. \end{aligned}$$

Solving the Black-Scholes PDE

Black-Scholes Model - Solving the PDE

- So far, we have shown that under our ansatz, option prices $\Pi_t = F(t, S_t)$ must satisfy the Black-Scholes PDE.
- The Black-Scholes PDE is an example of a boundary value problem. These problems have been studied in physics, where they are also known as parabolic partial differential equations. A special case is the heat equation, used to model the diffusion of heat through a medium.
- We will first provide a solution in terms of a new measure $\mathbb{Q}$, which we will call the risk-neutral measure.
- We will then show how to get this new measure $\mathbb{Q}$ from our physical measure $\mathbb{P}$.

Black-Scholes Model - Feynman-Kac Theorem

- The solution to the Black-Scholes PDE can be derived from the following theorem:

Theorem (Feynman-Kac Theorem)

If F is a solution to

$$F_t(t,x) + \mu(t,x)F_x(t,x) + \frac{1}{2}\sigma^2(t,x)F_{xx}(t,x) = rF(t,x)\,,$$
$$F(T,x) = \Phi(x)\,,$$

then F has the representation

$$F(t,x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)\,|\,X_t = x]\,,$$

where $\{X_t\}_{t\geq 0}$ satisfies

$$dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t^{\mathbb{Q}} \quad \textit{and } X_t = x\,.$$

Black-Scholes Model - Feynman-Kac Theorem

- We see that the Black-Scholes PDE is a special case of the boundary value problem in the Feynman-Kac theorem.
- Specifically, we make the substitutions

$$
\begin{aligned}
x &= S_t \,, \\
X_t &= S_t \,, \\
\mu(t,x) &= rS_t \,, \\
\sigma(t,x) &= \sigma S_t \,, \\
\text{“}\,|\, X_t = x\text{”} &= \text{“}\,|\,\mathcal{F}_t\text{”} \,.
\end{aligned}
$$

- The result of the Feynman-Kac theorem will give us the Risk-Neutral Valuation Formula.

Black-Scholes Model - Feynman-Kac Theorem

- The solution to the Black-Scholes PDE can be derived from the following theorem:

Theorem (Feynman-Kac Theorem, Black-Scholes Version)

If F is a solution to

$$F_t(t,S_t) + rS_t F_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{ss}(t,S_t) = rF(t,S_t)\,,$$
$$F(T,S_T) = \Phi(S_T)\,,$$

then F has the representation

$$F(t,S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)\,|\,\mathcal{F}_t]\,,$$

where $\{S_t\}_{t\geq 0}$ satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}\,.$$

Black-Scholes Model - Feynman-Kac Theorem

Proof.

We apply Ito's lemma to $e^{-rt}F(t, S_t)$:

$$
\begin{aligned}
de^{-rt}F &= (e^{-rt}F_t - re^{-rt}F)\,dt + e^{-rt}F_s\,dS_t + \frac{1}{2}e^{-rt}F_{ss}(dS_t)^2 \\
&= \dots \\
&= e^{-rt}\left(F_t - rF + rS_tF_s + \frac{1}{2}\sigma^2S_t^2F_{ss}\right) + e^{-rt}\sigma S_tF_s\,dW_t^{\mathbb{Q}} \\
&= e^{-rt}\sigma S_tF_s\,dW_t^{\mathbb{Q}}.
\end{aligned}
$$

□

Black-Scholes Model - Feynman-Kac Theorem

Proof (cont'd).

Taking the integral of both sides from t to T gives

$$e^{-rT}F(T,S_T) - e^{-rt}F(t,S_t) = \int_t^T e^{-ru}\sigma S_u F_s(u,S_u)\, dW_u^{\mathbb{Q}}\,.$$

Now take the conditional expectation with respect to $\mathcal{F}_t$ of this equation. Since the Ito integral is a martingale, the conditional expectation on the right side vanishes. We are left with

$$\begin{aligned} \mathbb{E}[e^{-rT}F(T,S_T)\,|\,\mathcal{F}_t] &= \mathbb{E}[e^{-rt}F(t,S_t)\,|\,\mathcal{F}_t] \\ e^{-r(T-t)}\mathbb{E}[F(T,S_T)\,|\,\mathcal{F}_t] &= F(t,S_t) \\ F(t,S_t) &= e^{-r(T-t)}\mathbb{E}[\Phi(S_T)\,|\,\mathcal{F}_t]\,. \end{aligned}$$

□

Black-Scholes Model - The Risk-Neutral Measure

- Now there is just one missing piece: how do we get the measure $\mathbb{Q}$?
- Recall that under the physical measure $\mathbb{P}$, we have

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}} \, .$$

- However, for this formula to work, we need a measure $\mathbb{Q}$ such that

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}} \, .$$

- Note that since α and r do not necessarily coincide, $\mathbb{P}$ and $\mathbb{Q}$ are not the same measure either.

Black-Scholes Model - The Risk-Neutral Measure

- Equating these two equations gives

$$\alpha S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}} = rS_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}}$$
$$dW_t^{\mathbb{Q}} = \left(\frac{\alpha - r}{\sigma} \right) dt + dW_t^{\mathbb{P}} .$$

- Hence, all we need is a measure $\mathbb{Q}$ under which $\int_0^t \left(\frac{\alpha - r}{\sigma} \right) du + W_t^{\mathbb{P}}$ is a Brownian motion.
- It turns out that such a measure does exist by the Girsanov theorem.

Black-Scholes Model - The Risk-Neutral Measure

- Therefore, the Girsanov theorem justifies the following definition:

Definition (Risk-Neutral Measure)

A measure $\mathbb{Q}$ is a risk-neutral measure or a martingale measure if

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}\,.$$

- Again, the term "risk-neutral measure" comes from the pricing valuation formula. There is no risk premium under $\mathbb{Q}$.

- The term "martingale measure" comes from the fact that the discounted stock price $\{S_t/B_t\}_{t\geq 0}$ as well as the discounted option price $\{\Pi_t/B_t\}_{t\geq 0}$ are martingales.

Black-Scholes Model - Resolving the Ansatz

- Finally, we can justify our initial ansatz. Everything we have done so far can be used to prove the following:

Theorem

Suppose that F satisfies the Black-Scholes PDE for a given payoff function Φ. Define the portfolio

$$\theta_t^S = F_s(t, S_t) \quad \text{and} \quad \theta_t^B = F(t, S_t)/B_t - S_t F_s(t, S_t)/B_t \,.$$

Then θ is self-financing and a replicating portfolio for $X = \Phi(S_T)$, and

$$V_t^\theta = \Pi_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[X \,\middle|\, \mathcal{F}_t\right],$$

where $\mathbb{Q}$ is a measure under which $dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}$, i.e. a risk-neutral measure.

- Note that as a corollary, the market is complete.

Black-Scholes Formulas

Black-Scholes Formulas - Setup

Theorem (Risk-Neutral Valuation)

Suppose the continuous-time model is arbitrage free. Then the risk-neutral price of a contingent claim $X = \Phi(S_T)$ at time $t \leq T$ is given by

$$\Pi_X(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[X \,\middle|\, \mathcal{F}_t\right],$$

where $\mathbb{Q}$ is a risk-neutral measure. That is,

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}.$$

- Therefore in order to find the price of a contingent claim, we need to evaluate the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \,|\, \mathcal{F}_t]$ for a given function Φ.
- Calculating this expectation is difficult in general. However, in the case of European calls and puts, a closed-form solution exists. These are the Black-Scholes formulas.

Black-Scholes Formulas - Setup

- Note that under the risk-neutral measure $\mathbb{Q}$, the stock price $\{S_t\}_{t \geq 0}$ is a GBM with drift rS_t.
- Therefore by the results from before, we have

$$S_T \,|\, \mathcal{F}_t \sim^{\mathbb{Q}} log\mathcal{N}\left(\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t),\, \sigma^2(T-t)\right)$$

$$\ln(S_T) \,|\, \mathcal{F}_t \sim^{\mathbb{Q}} \mathcal{N}\left(\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t),\, \sigma^2(T-t)\right).$$

- Recall that under the physical measure $\mathbb{P}$, we had an α term. Under the risk-neutral measure, α is replaced by r.

Black-Scholes Formulas - Setup

- To simplify notation, let $Z_T = \ln(S_T)$, $\tilde{\mu} = \ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T - t)$, and $\tilde{\sigma}^2 = \sigma^2(T - t)$.
- This gives the following:
$$Z_T \,|\, \mathcal{F}_t \sim^{\mathbb{Q}} \mathcal{N}\left(\tilde{\mu}, \tilde{\sigma}^2\right).$$
- Note that $Z_T \,|\, \mathcal{F}_t$ is normally distributed under $\mathbb{Q}$, which is why we use the letter Z.
- Furthermore, we have $S_T = e^{Z_T}$. Therefore
$$\mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \,|\, \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(e^{Z_T}\right) \,\middle|\, \mathcal{F}_t\right].$$

Black-Scholes Formulas - Setup

- Since $Z_T \,|\, \mathcal{F}_t$ is normally distributed, its density function is

$$f_{Z_T \,|\, \mathcal{F}_t}(z) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\tilde{\sigma}}\right)^2} .$$

- Hence, we have

$$\mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \,|\, \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(e^{Z_T}\right) \,\middle|\, \mathcal{F}_t\right] = \int_{-\infty}^{\infty} \Phi(e^z) \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\tilde{\sigma}}\right)^2} \, dz \, ,$$

$$\Pi_t = e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(e^z) \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\tilde{\sigma}}\right)^2} \, dz \, .$$

- Despite how nasty this integral looks, it is fairly straightforward to compute numerically using code.

Black-Scholes Formulas - Call Option

- Let us now compute the price of a call option. Recall that the contract function of a call option with strike price K is $\Phi(s) = \max\{s - K, 0\} = (s - K)_+$.

- It will be easier to write this contract function in terms of indicator functions. Recall that for a set A, the indicator function of A is

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

- Therefore we have

$$\begin{aligned} \Phi(s) = (s - K)_+ &= (s - K)\mathbb{1}_{\{s \geq K\}} \\ &= s\mathbb{1}_{\{s \geq K\}} - K\mathbb{1}_{\{s \geq K\}}. \end{aligned}$$

Black-Scholes Formulas - Call Option

- By writing the payoff function in terms of indicator functions, we have the following for the price of a call:

$$\begin{aligned}\Pi_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \,|\, \mathcal{F}_t] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} - K\mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] \\ &\quad - e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[K\mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] .\end{aligned}$$

- Note that we have split this conditional expectation into two terms.

Black-Scholes Formulas - Call Option

Lemma

Suppose that $S_T \,|\, \mathcal{F}_t \sim^{\mathbb{Q}} \log\mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$. *Let* $N(\cdot)$ *denote the cdf of the standard normal distribution. Then we have the following:*

1. $\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] = e^{\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}} N\left(\frac{\tilde{\mu} + \tilde{\sigma}^2 - \ln(K)}{\tilde{\sigma}}\right) = \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t]\, N\left(\frac{\tilde{\mu} + \tilde{\sigma}^2 - \ln(K)}{\tilde{\sigma}}\right)$,
2. $\mathbb{E}^{\mathbb{Q}}\left[K \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] = K\, N\left(\frac{\tilde{\mu} - \ln(K)}{\tilde{\sigma}}\right)$,
3. $\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T < K\}} \,\middle|\, \mathcal{F}_t\right] = e^{\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}} N\left(\frac{\ln(K) - \tilde{\mu} - \tilde{\sigma}^2}{\tilde{\sigma}}\right) = \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t]\, N\left(\frac{\ln(K) - \tilde{\mu} - \tilde{\sigma}^2}{\tilde{\sigma}}\right)$,
4. $\mathbb{E}^{\mathbb{Q}}\left[K \mathbb{1}_{\{S_T < K\}} \,\middle|\, \mathcal{F}_t\right] = K\, N\left(\frac{\ln(K) - \tilde{\mu}}{\tilde{\sigma}}\right)$.

Black-Scholes Formulas - Call Option

Proof.

Here, (1) is a particularly nasty calculation. (2) and (4) are much nicer, and (3) is easily obtained from (1).

To show (1), we write the expectation as an integral. Then, we use the substitution $y = \frac{z-\tilde{\mu}}{\tilde{\sigma}}$. We then complete the square in the exponent, then use the substitution $u = y - \tilde{\sigma}$ to recover the density of the standard normal. Simplifying gives the desired formula. □

Black-Scholes Formulas - Call Option

Proof (cont'd).

Defining $y_0 = \frac{\ln(K)-\tilde{\mu}}{\tilde{\sigma}}$, it goes something like this:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{Z_T} \mathbb{1}_{\{Z_T \geq \ln(K)\}} \,\middle|\, \mathcal{F}_t\right] \\
&= \int_{\ln(K)}^{\infty} e^z \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\tilde{\sigma}}\right)^2} dz \\
&= \int_{y_0}^{\infty} e^{\tilde{\mu}+\tilde{\sigma} y} \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \tilde{\sigma}\, dy = \int_{y_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2 - 2\tilde{\sigma} y - 2\tilde{\mu})} dy \\
&= e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}} \int_{y_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\tilde{\sigma})^2} dy = e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}} \int_{y_0-\tilde{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\
&= e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}} (1 - N(y_0 - \tilde{\sigma})) = e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}} N(\tilde{\sigma} - y_0) \\
&= e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}} N\left(\frac{\tilde{\mu} + \tilde{\sigma}^2 - \ln(K)}{\tilde{\sigma}}\right).
\end{aligned}
$$

□

Black-Scholes Formulas - Call Option

Proof (cont'd).

For (2), we can take the K out of the expectation. This gives

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[K\mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] &= K\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] \\
&= K\mathbb{Q}(S_T \geq K \,|\, \mathcal{F}_t) \\
&= K\mathbb{Q}(Z_T \geq \ln(K) \,|\, \mathcal{F}_t) \\
&= K\left(1 - N\left(\frac{\ln(K) - \tilde{\mu}}{\tilde{\sigma}}\right)\right) \\
&= K\,N\left(\frac{\tilde{\mu} - \ln(K)}{\tilde{\sigma}}\right).
\end{aligned}$$

Note that in (1), we cannot take the S_T out of the expectation! □

Black-Scholes Formulas - Call Option

Proof (cont'd).

For (3), we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T<K\}} \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[S_T(1-\mathbb{1}_{\{S_T\geq K\}}) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S_T \,|\, \mathcal{F}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T\geq K\}} \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S_T \,|\, \mathcal{F}_t\right] - \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t]\, N\left(\frac{\tilde{\mu}+\tilde{\sigma}^2-\ln(K)}{\tilde{\sigma}}\right) \\
&= \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t]\left(1-N\left(\frac{\tilde{\mu}+\tilde{\sigma}^2-\ln(K)}{\tilde{\sigma}}\right)\right) \\
&= \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t]\, N\left(\frac{\ln(K)-\tilde{\mu}-\tilde{\sigma}^2}{\tilde{\sigma}}\right).
\end{aligned}
$$

The proof of (4) is left as an exercise. □

Black-Scholes Formulas - Call Option

- Recall that the price of a call is

$$\Pi_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] - e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[K \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right].$$

- Let's simplify the first term...

Black-Scholes Formulas - Call Option

- We have

$$
\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T \geq K\}} \,\middle|\, \mathcal{F}_t\right] &= e^{-r(T-t)} e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}} N\left(\frac{\tilde{\mu}+\tilde{\sigma}^2-\ln(K)}{\tilde{\sigma}}\right) \\
&= e^{-r(T-t)} S_t e^{r(T-t)} N\left(\frac{\ln(S_t)+\left(r-\frac{\sigma^2}{2}+\sigma^2\right)(T-t)-\ln(K)}{\sigma\sqrt{T-t}}\right) \\
&= S_t\, N\left(\frac{\ln(S_t/K)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= S_t\, N(d_1)\,,
\end{aligned}
$$

where

$$
d_1 := \frac{\ln(S_t/K)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\,.
$$

Black-Scholes Formulas - Call Option

- The second term simplifies to

$$
\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[K\mathbb{1}_{\{S_T\geq K\}}\,\middle|\,\mathcal{F}_t\right] &= e^{-r(T-t)}K\,N\left(\frac{\tilde{\mu}-\ln(K)}{\tilde{\sigma}}\right)\\
&= Ke^{-r(T-t)}\,N\left(\frac{\ln(S_t)+\left(r-\frac{\sigma^2}{2}\right)(T-t)-\ln(K)}{\sigma\sqrt{T-t}}\right)\\
&= Ke^{-r(T-t)}\,N\left(\frac{\ln(S_t/K)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&= Ke^{-r(T-t)}\,N(d_2)\,,
\end{aligned}
$$

where

$$
d_2 := \frac{\ln(S_t/K)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}\,.
$$

Black-Scholes Formulas - Call Option

- We have proved the following:

Theorem (Black-Scholes Formula - Call Option)

In the Black-Scholes model, the arbitrage-free time-t price of a European call option with strike K and maturity T is

$$c_t = S_t\, N(d_1) - Ke^{-r(T-t)}\, N(d_2)\,,$$

where

$$d_1 := \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

Black-Scholes Formulas - Put Option

- The price of a put option can be proved in a similar manner, or by applying put-call parity (exercise):

Theorem (Black-Scholes Formula - Put Option)

In the Black-Scholes model, the arbitrage-free time-t price of a European put option with strike K and maturity T is

$$p_t = Ke^{-r(T-t)}\, N(-d_2) - S_t\, N(-d_1)\,,$$

where

$$d_1 := \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

Black-Scholes Formulas - Dividends

- So far, we have assumed that the stock S does not pay dividends.
- However, if the stock pays continuous dividends at a rate of δ, then our model will exhibit the following:

$$dS_t = (\alpha - \delta) S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}} \, ,$$
$$dS_t = (r - \delta) S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}} \, .$$

- Under the physical measure $\mathbb{P}$, α is replaced by $\alpha - \delta$. Under the risk-neutral measure, r is replaced by $r - \delta$ (in fact, this is how we define $\mathbb{Q}$).

Black-Scholes Formulas - Dividends

Theorem (Black-Scholes Formulas - Dividends)

In the Black-Scholes model, the arbitrage-free time-t prices of European options with strike K and maturity T are

$$c_t = S_t e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)\,,$$

$$p_t = K e^{-r(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1)\,,$$

where

$$d_1 := \frac{\ln(S_t/K) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

- Note that we have an extra factor of $e^{-\delta(T-t)}$ in the S_t term. This resembles the formula for put-call parity.
- Note also that we replace r with $r - \delta$ in the calculation of d_1. This resembles what we did with the risk-neutral probabilities in the binomial model.

Black-Scholes Formulas - Example

Example

A stock with current price $S_0 = 100$ is modeled using the Black-Scholes framework. You are given $r = 5\%$, $\sigma = 0.2$, and that the stock pays continuous dividends at a rate of 4%. Calculate the price of an at-the-money European call with expiry $T = 3$ months.

Black-Scholes Formulas - Example

Example

We calculate d_1 and d_2 first. This gives

$$d_1 = \frac{\ln(S_t/K) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = 0.075$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = -0.025\,.$$

We can then calculate the values of $N(d_1)$ and $N(d_2)$ from the standard normal table. This gives

$$N(d_1) \approx N(0.08) = 0.5319, \quad N(d_2) \approx 1 - N(0.03) = 0.4880\,,$$

where we round the values when using the standard normal table.

Black-Scholes Formulas - Example

Example

Plugging this into the B-S formula gives

$$\begin{aligned} c_0 &= S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) \\ &= 100 e^{-0.04/4}(0.5319) - 100 e^{-0.05/4}(0.4880) \\ &= \$4.47\,. \end{aligned}$$

Black-Scholes Formulas - Example

Example (Exercise)

Under the Black-Scholes framework, you are given a stock with current price \$40 that pays continuous dividends at a rate of 4%. The volatility of the stock is 40%, and the risk free rate is 4%.

The price of a 1-year put option with strike \$36 is the same as the price of a 1-year call option with strike K. Verify that K is approximately \$46. Is the true value of K greater than \$46?

The Greeks and Hedging

The Greeks - Introduction

- We have seen that the arbitrage-free price of a call option under the Black-Scholes framework is the given by

$$c_t = S_t e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \, .$$

- However, it is not immediately clear how the price of the call option depends on each of the different parameters in the model.
  - For example, we would expect that the price of the call increases as the stock price increases. If the stock price increases, it would be more likely to experience a larger payoff.
  - We may also expect that the option price increases as the volatility σ increases. Since options can be used as insurance, we expect the price of insurance to increase when the risk increases.

The Greeks - Introduction

- To answer these questions, we will take partial derivatives with respect to the parameters of the model.

Definition (The Greeks)

The Greeks are partial derivatives of the Black-Scholes formulas for European calls and puts, with respect to its parameters.

- The term “Greeks” comes from the fact that (some of) these quantities are denoted by Greek letters.

- There are many different Greeks used in practice. We will only introduce 5 different Greeks in this course.

The Greeks - Introduction

- Recall from Part I that we obtained some results on the behaviour of option prices with respect to changes in the strike K and changes in expiration T. These results still hold in the Black-Scholes framework.
- We will now focus on the remaining parameters. Note that for a given option, the strike K and the expiration T are fixed upon creation of the contract.
- However, over time, the values of t, S_t, σ, and r (to a lesser extent) can all change.

The Greeks - Delta

- The most important Greek is the Delta, denoted by the letter Δ.

Definition (Delta)

The delta is the first derivative of the option price with respect to the stock price. That is,

$$\Delta^{(c)} = \frac{\partial c}{\partial S} \quad \text{and} \quad \Delta^{(p)} = \frac{\partial p}{\partial S}.$$

- Hence, the delta measures the sensitivity of the option price with respect to the stock price.

The Greeks - Delta

- We can obtain closed-form solutions to many of the Greeks, including the Delta.
- It will be easy to derive the formulas just for the call option, and then apply put-call parity to obtain the corresponding formula for the put option.

Proposition

The deltas for European calls and puts are

$$\Delta^{(c)} = e^{-\delta(T-t)} N(d_1) \quad \textit{and} \quad \Delta^{(p)} = -e^{-\delta(T-t)} N(-d_1)\,.$$

The Greeks - Delta

Proof.

This is a somewhat difficult calculation, and requires evaluating the density of the normal distribution denoted by ϕ.

First, it is possible to verify that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}\,.$$

and also that

$$d_1^2 - d_2^2 = 2\ln(S_t e^{(r-\delta)(T-t)}/K)\,.$$

□

The Greeks - Delta

Proof.

We now have, for some expressions A_1 and A_2,

$$\begin{aligned}
\frac{\partial c}{\partial S} &= \frac{\partial}{\partial S}\left(S_t e^{-\delta(T-t)}\,N(d_1) - K e^{-r(T-t)}\,N(d_2)\right) \\
&= e^{-\delta(T-t)}\,N(d_1) + S_t e^{-\delta(T-t)}\,\phi(d_1)\frac{\partial d_1}{\partial S} - K e^{-r(T-t)}\,\phi(d_2)\frac{\partial d_2}{\partial S} \\
&= e^{-\delta(T-t)}\,N(d_1) + A_1\left(S_t e^{-\delta(T-t)}\,\phi(d_1) - K e^{-r(T-t)}\,\phi(d_2)\right) \\
&= e^{-\delta(T-t)}\,N(d_1) + A_2\underbrace{\left(1 - \frac{K e^{-r(T-t)}}{S_t e^{-\delta(T-t)}}\,e^{\frac{1}{2}(d_1^2-d_2^2)}\right)}_{=0} \\
&= e^{-\delta(T-t)}\,N(d_1)\,.
\end{aligned}$$

□

The Greeks - Delta

Proof.

For the put option delta, recall put-call parity:

$$c_t - p_t = S_t e^{-\delta(T-t)} - Ke^{-r(T-t)} .$$

Differentiating this with respect to S and rearranging gives

$$\begin{aligned}
\Delta^{(c)} - \Delta^{(p)} &= e^{-\delta(T-t)} \\
\Delta^{(p)} &= e^{-\delta(T-t)} N(d_1) - e^{-\delta(T-t)} \\
&= e^{-\delta(T-t)} (N(d_1) - 1) \\
&= -e^{-\delta(T-t)} N(-d_1) .
\end{aligned}$$

□

The Greeks - Delta

- Note that the delta for a call option is always between 0 and 1.
  - This implies that the price of a call option increases as the stock price increases. This matches what we expect.
  - This also means that the call price does not change as much as the stock price does.
- Correspondingly, the delta for a put option is always between -1 and 0.
  - This implies that the price of a put option decreases as the stock price increases. This matches what we expect.
  - This also means that the put price does not change as much as the stock price does.

The Greeks - Delta

- In practice, the (absolute value of) delta is often used as an approximation for the probability that the option will expire in-the-money.
- For example, if a put option has $\Delta^{(p)} = -0.6$, then it would have approximately a 60% chance of expiring in-the-money ($S_T < K$).
- This does not really have a theoretical basis. In fact, we have seen that the probability of a call expiring in-the-money is actually

$$\mathbb{Q}(S_T \geq K \,|\, \mathcal{F}_t) = N(d_2)\,.$$

- But the above is under the risk-neutral measure $\mathbb{Q}$. The physical probability measure $\mathbb{P}$ is a different story.

The Greeks - Gamma

Definition (Gamma)

The gamma is the second derivative of the option price with respect to the stock price. That is,

$$\Gamma^{(c)} = \frac{\partial^2 c}{\partial S^2} \quad \text{and } \Gamma^{(p)} = \frac{\partial^2 p}{\partial S^2}.$$

- Note that

$$\Gamma^{(c)} = \frac{\partial \Delta^{(c)}}{\partial S} \quad \text{and} \quad \Gamma^{(p)} = \frac{\partial \Delta^{(p)}}{\partial S}.$$

  That is, the gamma is the change in the delta induced by change in a stock price.

- We will see that the gamma will be important for hedging.

The Greeks - Gamma

Proposition

The gamma for European calls and puts is

$$\Gamma^{(c)} = \Gamma^{(p)} = \frac{e^{-\delta(T-t)}\phi(d_1)}{S_t\sigma\sqrt{T-t}}.$$

- Note that the gamma for both calls and puts are the same, and always greater than zero.
- This implies that the option value is convex in relation to the price of its underlying. This is different from the result in Part I!
- Roughly, gamma is largest when options are (approximately) at the money.

The Greeks - Vega

- We will note derive the formulas for the remaining Greeks. The derivations are similar to that of the delta.

Definition (Vega)

The vega is the derivative of the option price with respect to the volatility σ. That is,

$$\nu^{(c)} = \frac{\partial c}{\partial \sigma} \quad \text{and} \quad \nu^{(p)} = \frac{\partial p}{\partial \sigma}.$$

The formula for the vega of European calls and puts is

$$\nu^{(c)} = \nu^{(p)} = S_t e^{-\delta(T-t)} \sqrt{T-t}\, \phi(d_1).$$

- Note that the vega for calls and puts is the same, and always positive. This implies that the price of the options increase as volatility increases.

The Greeks - Theta

Definition (Theta)

The theta is the derivative of the option price with respect to time t. That is,

$$\Theta^{(c)} = \frac{\partial c}{\partial t} \quad \text{and} \quad \Theta^{(p)} = \frac{\partial p}{\partial t}.$$

The formulas are

$$\Theta^{(c)} = \delta S_t e^{-\delta(T-t)} N(d_1) - rKe^{-r(T-t)} N(d_2) - \frac{Ke^{-r(T-t)}\sigma\,\phi(d_2)}{2\sqrt{T-t}},$$

$$\Theta^{(p)} = \Theta^{(c)} - \delta S_t e^{-\delta(T-t)} + rKe^{-r(T-t)}.$$

- It can be verified that $\Theta^{(c)} < 0$, which implies that the call price decreases with the passage of time. However, it is possible to have $\Theta^{(p)} < 0$ and $\Theta^{(p)} > 0$.

The Greeks - Rho

Definition (Rho)

The rho is the derivative of the option price with respect to the risk-free rate r. That is,

$$\rho^{(c)} = \frac{\partial c}{\partial r} \quad \text{and} \quad \rho^{(p)} = \frac{\partial p}{\partial r}\,.$$

The formulas are

$$\rho^{(c)} = (T-t)Ke^{-r(T-t)}N(d_2)\,,$$
$$\rho^{(p)} = -(T-t)Ke^{-r(T-t)}N(-d_2)\,.$$

- We have $\rho^{(c)} > 0$ and $\rho^{(p)} < 0$. Generally (short-term) interest rates do not vary too much. However, the ρ may be of more interest in the current financial climate.

Hedging - Introduction

- Just as we have defined the Greeks for the options, we can also define the Greeks of a portfolio.
- Suppose we have N assets, denoted $S^{(1)}, \ldots, S^{(N)}$. A portfolio in this market is a column vector $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \ldots & \theta_N \end{bmatrix}^T$.
- The value of this portfolio is

$$V_t^\theta = S_t \cdot \theta = \sum_{i=1}^{N} \theta_i S_t^{(i)} .$$

- Suppose that each asset has the same underlying instrument S. We can define the delta of a portfolio at time t as

$$\Delta_t^\theta := \frac{\partial}{\partial S} V_t^\theta .$$

- The other Greeks of a portfolio are defined in a similar manner.

Hedging - Introduction

- Since the partial derivative is a linear operator, we have

$$\frac{\partial}{\partial S} V_t^{\theta} = \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial S} S_t^{(i)} = \sum_{i=1}^{N} \theta_i \Delta^{(i)} .$$

- Hence, the Greeks for a portfolio is the weighted sum of the Greeks of the portfolio's constituent assets.
- Recall that in Part II, we denoted a portfolio at time t by (Δ_t, b_t). It is easy to verify that Δ_t is the delta of this portfolio at time t.

Hedging - Definitions

Definition (Hedging, *Wikipedia* Version)

A hedge is an investment position intended to offset potential losses or gains that may be incurred by a companion investment. A hedge can be constructed from many types of financial instruments, including stocks, forwards, options, etc.

- For our purposes, we will use the following (slightly more mathematical) definition:

Definition (Hedging)

A hedge is a portfolio θ constructed in a manner that controls the impact of changes in other model parameters (e.g. S_t, σ, etc.) on the value of the portfolio V_t^{θ}.

Hedging - Delta-Hedging

- In essence, hedging means constructing portfolios while controlling the value of the Greeks. While a hedge can in theory have an arbitrary target for the Greeks, the most common types of hedging are the following:

Definition (Delta-Hedged)

A portfolio θ is said to be delta-hedged or delta-neutral if its delta is zero. That is,

$$\Delta^{\theta} = \sum_{i=1}^{N} \theta_i \Delta^{(i)} = 0 \,.$$

- This means that the value of the portfolio will not change as a result of small changes in the price of the underlying asset S.

Example

Assume the Black-Scholes framework, with $r = 6\%$ and $\sigma = 0.25$. A stock with current price $S_0 = 200$ pays continuous dividends at a rate of $\delta = 3\%$.

Suppose you sold a call option on the stock with strike $K = 210$ and time to expiry $T = 6$ months. You wish to hedge this position so that your resulting portfolio is delta-neutral. How many shares of the stock must you buy or sell?

Hedging - Delta-Hedging

Example

Let θ^S denote the number of shares of the stock. In order for the portfolio to be delta-neutral, we must have

$$0 = \Delta^\theta = -1 \times \Delta^{(c)} + \theta^S \times \Delta^{(S)} .$$

Note that the delta of a stock is 1. Therefore, rearranging the above gives

$$\theta^S = \Delta^{(c)} = e^{-\delta(T-t)} N(d_1) .$$

As an exercise, verify that the value of θ^S is ≈ 0.4533, implying that you must buy 0.4533 shares of stock to delta-hedge this position.

Hedging - Delta-Hedging

- The value of a delta-hedged portfolio does not change as a result of "small" changes in the price of the underlying asset S.
- However, this is complicated by the fact that Δ_t^θ changes as t progresses and as S_t changes.
- Hence, in order to obtain a truly delta-neutral position, the portfolio must be continuously rebalanced.
- In fact, when developing the Black-Scholes model, we constructed a continuously rebalanced delta-neutral portfolio. We used $\theta_t^S = F_s(t, S_t)$ as the weight of the stock in the portfolio, which is exactly the delta of the option!
- This process is referred to as dynamic hedging.

Hedging - Delta-Gamma-Hedging

- Another way to look at hedging is through Taylor series. If $V^\theta(S_t)$ is the value of a portfolio θ as a function of stock price, then the Taylor expansion is

$$\begin{aligned} V^\theta(S_t+\varepsilon) &= V^\theta(S_t) + \varepsilon\frac{\partial V^\theta}{\partial S} + \frac{\varepsilon^2}{2}\frac{\partial^2 V^\theta}{\partial S^2} + \frac{\varepsilon^3}{6}\frac{\partial^3 V^\theta}{\partial S^3} + \dots \\ &= V^\theta(S_t) + \varepsilon\Delta^\theta + \frac{\varepsilon^2}{2}\Gamma^\theta + \frac{\varepsilon^3}{6}\frac{\partial^3 V^\theta}{\partial S^3} + \dots \end{aligned}$$

- Our goal is so that the value of the portfolio does not change as the price of the underlying changes. That is, we would like $V^\theta(S_t+\varepsilon) \approx V^\theta(S_t)$.
- Note that delta-hedging partially achieves this by setting $\Delta^\theta = 0$. However, we would do a better job if we also set $\Gamma^\theta = 0$.

Hedging - Delta-Gamma-Hedging

- This motivates the following definition:

Definition (Delta-Gamma-Hedged)

A portfolio θ is said to be delta-gamma-hedged or delta-gamma-neutral if its delta and gamma are zero. That is,

$$\Delta^{\theta} = \Gamma^{\theta} = 0 \, .$$

- It is difficult to continuously rebalance portfolios in practice. Delta-gamma-hedged portfolios partially alleviate this concern by providing a better hedge that may not need to be rebalanced as much.
- Note that while the delta of a stock is 1, the gamma of a stock is 0. This means that a stock cannot be used to change the gamma of a portfolio. Instead, we will have to use options, as shown in the next example.

Hedging - Delta-Gamma-Hedging

Example

Suppose you observe the following two different call options in the market, with the same underlying S:

Call option	Δ	Γ
$c^{(1)}$	0.541	0.1050
$c^{(2)}$	0.672	0.0822

Suppose you have sold one contract of $c^{(1)}$ and would like to delta-gamma-hedge your position using $c^{(2)}$ and S. Determine the required weights of each asset.

Calibrating a Binomial Tree

Calibrating a Binomial Tree - Introduction

- We have mentioned that the continuous Black-Scholes model and the discrete binomial model are related.
- Now we can investigate their relationship in slightly more detail. Specifically, given a Black-Scholes setting with the dynamics

$$\begin{aligned} dB_t &= rB_t\,dt \\ dS_t &= (\alpha - \delta)S_t\,dt + \sigma S_t\,dW_t^{\mathbb{P}} \\ &= (r - \delta)S_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}\,, \end{aligned}$$

we will show the following:

1. The Black-Scholes model is a limiting case of the binomial model.
2. Given the parameters of a Black-Scholes model, we can calibrate a binomial model so that it approximates the Black-Scholes model.

Calibrating a Binomial Tree - Black-Scholes as a Limit

- First recall that under the Black-Scholes model and its risk-neutral measure $\mathbb{Q}$, we have

$$S_T = S_0 e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T+\sigma W_T^{\mathbb{Q}}}.$$

- It will be useful to express the above in terms of the logarithmic return. Rearranging gives

$$\ln(S_T/S_0) = \left(r - \delta - \frac{\sigma^2}{2}\right) T + \sigma W_T^{\mathbb{Q}} \sim^{\mathbb{Q}} \mathcal{N}\left(\left(r - \delta - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right).$$

Calibrating a Binomial Tree - Black-Scholes as a Limit

- Now let us calculate the (limiting) log-return of the multiperiod model, under its risk-neutral measure $\mathbb{Q}$. Suppose a model divides a time period of length T into k periods of length h (i.e. $T = kh$).
- Then we have

$$\begin{aligned} \ln(S_T/S_0) &= \ln\left(\frac{S_T}{S_{T-h}} \times \frac{S_{T-h}}{S_{T-2h}} \times \cdots \times \frac{S_h}{S_0}\right) \\ &= \sum_{j=1}^{k} \ln\left(\frac{S_{jh}}{S_{(j-1)h}}\right), \end{aligned}$$

where $\ln\left(\frac{S_{jh}}{S_{(j-1)h}}\right)$ is the log-return over the j-th period.

Calibrating a Binomial Tree - Black-Scholes as a Limit

- However, we also know that the log-returns are i.i.d. with mean

$$m := q_u \ln(u) + q_d \ln(d)\,,$$

and variance

$$s^2 := q_u q_d (\ln(u) - \ln(d))^2\,.$$

- Applying the Central Limit Theorem to the log-return as $k \to \infty$ gives

$$\sqrt{k}\left(\frac{\ln(S_T/S_0)/k - m}{s}\right) \sim^{\mathbb{Q}} \mathcal{N}(0,1)\,.$$

Calibrating a Binomial Tree - Black-Scholes as a Limit

- Rearranging this expression gives

$$\sqrt{k}\left(\frac{\ln(S_T/S_0)/k - m}{s}\right) \sim^{\mathbb{Q}} \mathcal{N}(0,1)$$

$$\ln(S_T/S_0) \sim^{\mathbb{Q}} \mathcal{N}(km, ks^2)$$

$$\ln(S_T/S_0) \sim^{\mathbb{Q}} \mathcal{N}\left(\frac{m}{h}T, \frac{s^2}{h}T\right).$$

- This looks very similar to the log-returns under the Black-Scholes model. Note that as $k \to \infty$ (or, equivalently, as $h \to 0$), the terminal stock price S_T is lognormally distributed.

Calibrating a Binomial Tree - Choosing u and d

- Calibrating a binomial tree is essentially choosing the right values of u and d that approximate a Black-Scholes model.
- From what we have seen before, we would like to choose u and d in such a way such that

$$\lim_{h\to 0} \frac{m}{h} = r - \delta - \frac{\sigma^2}{2}\,,$$

$$\lim_{h\to 0} \frac{s^2}{h} = \sigma^2\,.$$

- It looks like this is a system of two equations and two unknowns (u and d), so we should be able to solve uniquely for u and d. However, it turns out that these two equations are actually the same equation under the risk neutral measure!

Calibrating a Binomial Tree - Choosing u and d

- Therefore we only need u and d to solve one of these equations. The second equation is simpler:

$$\lim_{h \to 0} \frac{s^2}{h} = \sigma^2 .$$

  Hence, u and d are chosen so that they approximate the volatility.

- Since there is only one equation and two unknowns, there is some flexibility in the choice of u and d.

Calibrating a Binomial Tree - Choosing u and d

- If we impose the additional condition $ud = 1$, then we have the following result.

Proposition (Standard Binomial Tree)

Let $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$. Then

$$\lim_{h \to 0} \frac{m}{h} = \lim_{h \to 0} \frac{q_u \ln(u) + q_d \ln(d)}{h} = r - \delta - \frac{\sigma^2}{2},$$

$$\lim_{h \to 0} \frac{s^2}{h} = \lim_{h \to 0} \frac{q_u q_d (\ln(u) - \ln(d))^2}{h} = \sigma^2.$$

and hence this binomial tree approximates a Black-Scholes model with volatility σ.

- It can be shown that $\lim_{h \to 0} q_u = \lim_{h \to 0} q_d = 1/2$ using l'Hôpital's rule. The result then follows fairly easily.

Calibrating a Binomial Tree - Choosing u and d

- If we impose the additional condition $q_u = q_d = 1/2$, then we have the following result.

Proposition

Let $u = e^{(r-\delta)h}(1 + \sqrt{e^{\sigma^2 h} - 1})$ *and* $d = e^{(r-\delta)h}(1 - \sqrt{e^{\sigma^2 h} - 1})$. *Then*

$$\lim_{h \to 0} \frac{m}{h} = \lim_{h \to 0} \frac{q_u \ln(u) + q_d \ln(d)}{h} = r - \delta - \frac{\sigma^2}{2},$$

$$\lim_{h \to 0} \frac{s^2}{h} = \lim_{h \to 0} \frac{q_u q_d (\ln(u) - \ln(d))^2}{h} = \sigma^2.$$

and hence this binomial tree approximates a Black-Scholes model with volatility σ.

Calibrating a Binomial Tree - Summary

- We see that the Black-Scholes model is indeed obtained as a limiting case of the binomial model.
- In practice, it might be preferable to use a binomial model instead of the Black-Scholes model.
- The "standard binomial tree" according to many textbooks is the one given by

$$u = e^{\sigma\sqrt{h}} \quad \text{and} \quad d = e^{-\sigma\sqrt{h}}.$$

- Hence, if you want to create a binomial tree model for a market, all you need to do is to choose a time period h and to estimate the volatility σ.

The Implied Volatility

The Implied Volatility - Introduction

- We have seen that under the Black-Scholes setting:

$$
\begin{aligned}
dB_t &= rB_t\, dt \\
dS_t &= (\alpha - \delta)S_t\, dt + \sigma S_t\, dW_t^{\mathbb{P}} \\
&= (r - \delta)S_t\, dt + \sigma S_t\, dW_t^{\mathbb{Q}} \\
c_t &= S_t e^{-\delta(T-t)}\, N(d_1) - Ke^{-r(T-t)}\, N(d_2)\,.
\end{aligned}
$$

- Suppose you are an investor observing the market at some time t. Note that you can observe the values of r, δ, S_t, K, t, T. However, you cannot observe α or σ!
- The parameter α does not appear in the pricing formulas (or the Black-Scholes PDE), so it does not matter too much. In any case, its maximum likelihood estimate (MLE), given $\mathcal{F}_t$, is

$$
\hat{\alpha} = \frac{\ln(S_t/S_0)}{t} + \delta\,.
$$

The Implied Volatility - Introduction

- The fact that volatility cannot be observed represents a more significant problem. Typically, there are two approaches to dealing with this:

(1) We can use past data to estimate volatility. This is known as historical volatility. Unfortunately, this process comes with several drawbacks.

(2) Instead, what is more common in practice is to use implied volatility.

Historical Volatility

- Recall that in the Black-Scholes framework,

$$d\ln(S_t) = \left(r - \delta - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t^{\mathbb{Q}}$$

$$\ln(S_t) \sim^{\mathbb{Q}} \mathcal{N}\left(\ln(S_0) + \left(r - \delta - \frac{\sigma^2}{2}\right) t,\ \sigma^2 t\right).$$

- Note that σ^2 is the variance of one-year logarithmic returns. This can be calculated from past data from what we know about the Brownian motion.

- If we had a list of one-year log-returns of the stock, we can calculate the variance of returns to estimate σ^2.

Historical Volatility

- However, there is often a lack of representative data for this process. Even for companies that have been around for 20 years, having 20 data points for estimation is not very convincing.
- Instead, we can estimate the variance over a smaller time frame, and then scale up to obtain an estimate for σ^2.
- This process is typically done with daily returns. There are an average of 252 trading days in a year, so letting $t = \frac{1}{252}$ gives

$$\ln(S_{1/252}) \sim^{\mathbb{Q}} \mathcal{N}\left(\ln(S_0) + \frac{1}{252}\left(r - \delta - \frac{\sigma^2}{2}\right), \sigma^2/252\right).$$

Historical Volatility

- Specifically, given daily (closing) prices of a stock S, we do the following:
  1. We take the log of each price to get observations for $\ln(S_{1/252})$.
  2. Calculate the variance of these observations. This gives us an estimate for $\sigma^2/252$.
  3. Scale this value back up (by multiplying by 252) to get an estimate for σ^2.
- This process is not entirely satisfying, especially since historical volatility calculated using different kinds of returns gives different values.
- Furthermore, historical volatility may not be an accurate predictor of future volatility. Recall that σ is actually the expected future volatility over the lifetime of the option.

Implied Volatility

- However, notice that even though the volatility cannot be observed, the price of calls and puts are observable in the market.
- Hence, we can essentially invert the Black-Scholes formula, and solve for the volatility given the price of an option.
- This is the idea behind implied volatility:

Definition (Implied Volatility, *Wikipedia* Version)

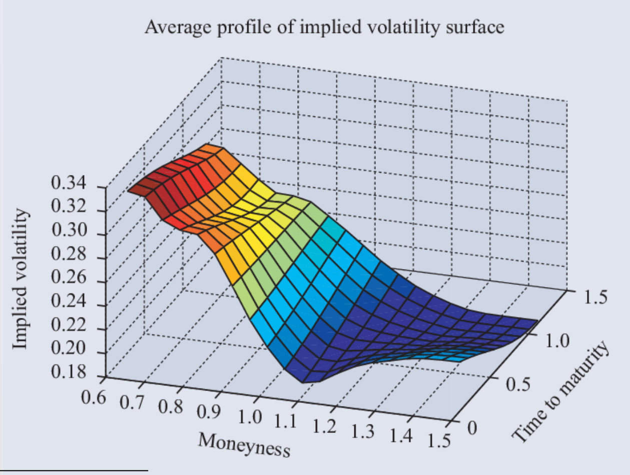
The implied volatility of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model (usually Black–Scholes), will return a theoretical value equal to the price of said option.

Implied Volatility

- The implied volatility is the market's attitudes towards the volatility of a stock, as reflected by market prices of options on this stock.
- Since inverting the Black-Scholes formulas are difficult, this is done using numerical methods.
- Under the Black-Scholes model, the implied volatility should be constant, regardless of the strike K and the expiration date T.
- However, in practice, the implied volatility exhibits a volatility smile when plotted against strike prices. This means that implied volatility is higher for options that are either deep in-the-money or deep out-of-the-money.

Implied Volatility Surface

- Below is a plot of implied volatility against strike and expiration, also known as an implied volatility surface:



[1]

[1] Image from Cont and Foncesca (2002), Dynamics of Implied Volatility Surfaces.

Monte-Carlo Pricing

Monte-Carlo Pricing - Introduction

- We have seen that under the Black-Scholes framework, pricing an option with payoff $X = \Phi(S_T)$ relies on calculating the discounted expectation under the risk-neutral measure:
$$\Pi_X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)]\,.$$
- It turns out that the same result holds for options that are not of the form $X = \Phi(S_T)$, including path-dependent options (e.g. Asian, lookback, etc).
$$\Pi_X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[X]\,.$$
- To calculate the prices of these options, we can use the Monte-Carlo method to numerically estimate the expectation $\mathbb{E}^{\mathbb{Q}}[X]$.

Monte-Carlo Pricing - Introduction

- The main idea is that we want to simulate many different i.i.d. paths of the stock price $\{S_t\}_{0 \leq t \leq T}$. Then, we can compute the option payoff X in every case.
- Then by the Law of Large Numbers, we should have

$$\mathbb{E}^{\mathbb{Q}}[X] \approx \frac{1}{n} \sum_{i=1}^{n} X(i),$$

where $X(i)$ denotes the payoff of the option in the i-th trial.

Monte-Carlo Pricing - Simulating a Stock Price

- It turns out that simulating a stock price is fairly straightforward. First, discretize the time period into periods of length h. Then we have

$$S_{t+h} = S_t e^{\left(r-\delta-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}Z},$$

  where $Z \sim \mathcal{N}(0,1)$.

- Using this formula, we can create a stock price process $\{S_0, S_h, S_{2h}, \ldots, S_T\}$ from i.i.d. standard normal samples.

- We then estimate the price as

$$\Pi_X(0) = e^{-rT}\frac{1}{n}\sum_{i=1}^{n} X(i).$$

Monte-Carlo Pricing - Summary

- Monte-Carlo methods can be used to handle more complicated derivatives, and even more complicated models than the Black-Scholes model.
- This process is quite efficient in practice.
- By adjusting parameters of the simulation, the Greeks for more complicated options can also be estimated.
- However, American options require special treatment, and are not straightforward to price using Monte-Carlo.